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Microlocalization of Topological Boundary Value Morphism and Regular-Specializable Systems

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Introduction

In microlocal analysis, it is one of the main subjects to give an appropriate formulation of the boundary value problems for hyperfunction or microfunction solutions to a system of linear partial differential equations with analytic coefficients (that is, a coherent (left) $\mathcal{D}$-Module, here in this article, we shall write *Module* with a capital letter, instead of *sheaf of modules*). If the system is *regular-specializable*, the *nearby-cycle* of the system can be defined in the theory of $\mathcal{D}$-Modules. After the results by Kashiwara and Oshima [K-O], Oshima [Os] and Schapira [Sc 2], [Sc 3], for any hyperfunction solutions to regular-specializable system Monteiro Fernandes [MF 1] defined a boundary value morphism which takes values in hyperfunction solutions to the nearby-cycle of the system instead of the *induced* system. This morphism is injective (cf. [MF 2]) and a generalization of the non-characteristic boundary value morphism (for the non-characteristic case, see Komatsu and Kawai [Ko-K], Schapira [Sc 1] and further Kataoka [Kat]). Moreover recently Laurent and Monteiro Fernandes [L-MF 2] reformulated this boundary value morphism and discussed the solvability under a kind of hyperbolicity condition (the *near-hyperbolicity*). However, since this morphism is defined only for hyperfunction solutions, a microlocal boundary value problem is not considered. Therefore in this article, we shall state a microlocalization of their result in the framework of Oaku [Oa 2] and Oaku-Yamazaki [O-Y].

The details of this article will be given in our forthcoming paper [Y].

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1 Notation

We denote the set of integers, of real numbers and of complex numbers by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ respectively as usual. Moreover we set $\mathbb{N} := \{n \in \mathbb{Z};\ n \geqslant 1\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

All the manifolds are assumed to be paracompact. Let $\tau\colon E \to Z$ a vector bundle over a manifold Z. Then, set $\dot{E} := E \setminus Z$ and $\dot{\tau}$ the restriction of τ to $\dot{E}$. Let M be an $(n+1)$-dimensional real analytic manifold and N a one-codimensional closed real analytic submanifold of M. Let X and Y be complexifications of M and N respectively such that Y is a closed submanifold of X and that $Y \cap M = N$. Moreover, we assume the existence of a partial complexification of M in X; that is, there exists a $(2n+1)$-dimensional real analytic submanifold L of X containing both M and Y such that the triplet (N, M, L) is locally isomorphic to $(\mathbb{R}^n \times \{0\}, \mathbb{R}^{n+1}, \mathbb{C}^n \times \mathbb{R})$ by a local coordinate system $(z,\tau) = (x + \sqrt{-1}\,y, t + \sqrt{-1}\,s)$ of X around each point of N. We say such a coordinate system *admissible*. We shall mainly follow the notation in Kashiwara-Schapira [K-S]; we denote the normal deformations of N and Y in M and L by $\widetilde{M}_N$ and $\widetilde{L}_Y$ respectively and regard $\widetilde{M}_N$ as a closed submanifold of $\widetilde{L}_Y$. We have the following commutative diagram:

$$\begin{tikzcd}
T_NM \arrow[rr, hook, "s_M"] \arrow[dr, "\tau_N"] \arrow[dd, hook] & & \widetilde{M}_N \arrow[dr, "p_M"] \arrow[dd] & & \Omega_M \arrow[ll, hook', "j_M"'] \arrow[dl, "\tilde{p}_M"'] \arrow[dd] & \\
& N \arrow[rr, hook] \arrow[dd, hook] & & M \arrow[rr, hook, "i_M"] \arrow[dd, hook] & & X \arrow[dd, equal] \\
T_YL \arrow[rr, "s_L"] \arrow[dr, "\tau_Y"] & & \widetilde{L}_Y \arrow[dr, "p_L"] & & \Omega_L \arrow[ll, "j_L"'] \arrow[dl, "\tilde{p}_L"'] & \\
& Y \arrow[rr, hook, "i_Y"] & & L \arrow[rr, hook, "i_L"] & & X,
\end{tikzcd}$$

and by admissible coordinates we have locally the following relation:

$$\begin{tikzcd}
N = \mathbb{R}^n_x \times \{0\} \arrow[r, hook] \arrow[d, hook] & M = \mathbb{R}^n_x \times \mathbb{R}_t \arrow[d, hook, "i"'] \arrow[dr, hook, "i_M"] & \\
Y = \mathbb{C}^n_z \times \{0\} \arrow[r, hook, "i_Y"] & L = \mathbb{C}^n_z \times \mathbb{R}_t \arrow[r, hook, "i_L"] & X = \mathbb{C}^n_z \times \mathbb{C}_\tau .
\end{tikzcd}$$

With these coordinates, we often identify T_YX and T_YL with X and L respectively.

The projection $\tau_Y\colon T_YL \to Y$ and $s_L\colon T_YL \to \widetilde{L}_Y$ induce natural mappings:

$$T^*_NY \xleftarrow[\tau_{Y\pi}]{} T_NM \underset{N}{\times} T^*_NY \xrightarrow[{}^t\tau'_Y]{\sim} T^*_{T_NM}T_YL \xleftarrow[{}^ts'_L]{\sim} T_NM \underset{\widetilde{M}_N}{\times} T^*_{\widetilde{M}_N}\widetilde{L}_Y \xrightarrow[s_{L\pi}]{} T^*_{\widetilde{M}_N}\widetilde{L}_Y ,$$

and by these mappings, we identify $T^*_{T_NM}T_YL$ with $T_NM \underset{N}{\times} T^*_NY$ and $T_NM \underset{\widetilde{M}_N}{\times} T^*_{\widetilde{M}_N}\widetilde{L}_Y$.

$T_Y L \setminus T_Y Y$ has two components with respect to its fiber. We denote one of them by $T_Y L^+$ and represent (at least locally) by fixing an admissible coordinate system

$$T_Y L^+ = \{(z,t) \in T_Y L;\ t > 0\}.$$

Moreover set $T_N M^+ := T_Y L^+ \cap T_N M$. Set an open embedding $f\colon T_Y L^+ \hookrightarrow T_Y L$ and $f_N := f|_{T_N M^+}\colon T_N M^+ \hookrightarrow T_N M$. We regard $T_N M^+ \underset{N}{\times} T_N^* Y$ as an open set of $T^*_{T_N M} T_Y L$. Moreover f induces mappings:

$$\begin{tikzcd} T^*_{T_N M^+} T_Y L^+ & T_N M^+ \underset{T_N M}{\times} T^*_{T_N M} T_Y L \arrow[l, "\sim"'] \arrow[r, "f_\pi"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & T^*_{T_N M} T_Y L \arrow[d, "\wr"] \\ & T_N M^+ \underset{N}{\times} T_N^* Y \arrow[r, "f_N \times \mathrm{id}"] & T_N M \underset{N}{\times} T_N^* Y. \end{tikzcd}$$

Hence we identify $T^*_{T_N M^+} T_Y L^+$ with $T_N M^+ \underset{N}{\times} T_N^* Y$, and f_π with $f_N \times \mathrm{id}$.

Let $\pi_{N,M}\colon T^*_{\widetilde{M}_N} \widetilde{L}_Y \to \widetilde{M}_N$ and $\pi_{N|M}\colon T^*_{T_N M} T_Y L \to T_N M$, be the natural projections. We denote as usual by ν and μ the Sato specialization and microlocalization functors respectively.

2 General Boundary Values

By using an admissible coordinate system we define a continuous section $\sigma\colon Y \to \dot{T}_Y X$ by $z \mapsto (z,1)$. Similarly we define ${}^t\sigma\colon Y \to \dot{T}^*_Y X$ by $z \mapsto (z,1)$. In general, let Z be a complex manifold, $\tau\colon E \to Z$ a complex vector bundle. Then, denote by $\mathbf{D}^b_{\mathbb{C}^\times}(E)$ the subcategory of $\mathbf{D}^b(E)$ consisting of $\mathbb{C}^\times$-conic objects.

2.1 Theorem. *For any object $\mathcal{F}$ of $\mathbf{D}^b(X)$ such that $\nu_Y(\mathcal{F}) \in \mathrm{Ob}\big(\mathbf{D}^b_{\mathbb{C}^\times}(T_Y X)\big)$, there exists the following natural isomorphism*:

$$f_\pi^{-1} \mu_{T_N M}(\nu_Y(i_L^! \mathcal{F})) \xrightarrow{\sim} f_\pi^{-1}\, \tau_{Y\pi}^{-1}\, \mu_N(\sigma^{-1}\, \nu_Y(\mathcal{F})) \otimes \omega_{L/X}\,.$$

2.2 Definition. For any object $\mathcal{F}$ of $\mathbf{D}^b(X)$ such that $\nu_Y(\mathcal{F}) \in \mathrm{Ob}\big(\mathbf{D}^b_{\mathbb{C}^\times}(T_Y X)\big)$, we define by virtue of Kashiwara-Schapira [K-S] and Theorem 2.1:

$$\begin{aligned} \beta\colon f_\pi^{-1}\, s_{L\pi}^{-1}\, \mu_{\widetilde{M}_N}(\boldsymbol{R} j_{L*} \widetilde{p}_L^{-1}\, i_L^! \mathcal{F}) &\to f_\pi^{-1} \mu_{T_N M}(\nu_Y(i_L^! \mathcal{F})) \\ &\xrightarrow{\sim} f_\pi^{-1}\, \tau_{Y\pi}^{-1}\, \mu_N(\sigma^{-1}\, \nu_Y(\mathcal{F})) \otimes \omega_{L/X}\,. \end{aligned}$$

2.3 Definition (Laurent-Monteiro Fernandes [L-MF 2]). We say an object $\mathcal{F}$ of $\mathbf{D}^b(X)$ is *near-hyperbolic* at $x_0 \in N$ (in dt-codirection) if there exist positive constants C and ε_1 such that

$$\begin{aligned} &\mathrm{SS}(\mathcal{F}) \cap \{(z,\tau;z^*,\tau^*) \in T^*X;\ |z - x_0|,\ |\tau| < \varepsilon_1,\ 0 < \mathrm{Re}\,\tau\} \\ &\qquad \subset \{(z,\tau;z^*,\tau^*) \in T^*X;\ |\mathrm{Re}\,\tau^*| < C\big(|\mathrm{Im}\,z^*|(|\mathrm{Im}\,z| + |\mathrm{Im}\,\tau|) + |\mathrm{Re}\,z^*|\big)\} \end{aligned}$$

holds by an admissible coordinate system. Here $\mathrm{SS}(\mathcal{F})$ denotes the *microsupport* of $\mathcal{F}$.

2.4 Theorem. *Let $\mathcal{F}$ be a object of $\mathbf{D}^b(X)$. Assume that $\nu_Y(\mathcal{F}) \in \mathrm{Ob}\big(\mathbf{D}^b_{\mathbb{C}^\times}(T_Y X)\big)$ and $\mathcal{F}$ is near-hyperbolic at $x_0 \in N$. Then, for any $p^* \in T^*_{T_N M^+} T_Y L^+$*

$$\beta\colon s_{L\pi}^{-1}\,\mu_{\widetilde{M}_N}(\boldsymbol{R}j_{L*}\,\widetilde{p}_L^{-1}\, i_L^!\,\mathcal{F})_{p^*} \to \mu_N(\sigma^{-1}\,\nu_Y(\mathcal{F}))_{\tau_{Y_\pi}(p^*)} \otimes \omega_{L/X}$$

is an isomorphism.

3 Regular-Specializable Systems

In this section, we shall recall the basic results concerning the regular-specializable $\mathcal{D}$-Module and its nearby-cycle.

As usual, we denote by $\mathcal{D}_X$ the sheaf on X of holomorphic differential operators, and by $\{\mathcal{D}_X^{(m)}\}_{m\in\mathbb{N}_0}$ the usual order filtration on $\mathcal{D}_X$.

3.1 Definition. Denote by $\mathcal{I}_Y$ the defining Ideal of Y in $\mathcal{O}_X$ with a convention that $\mathcal{I}_Y^j = \mathcal{O}_X$ for $j \leqslant 0$. The *V-filtration* $\{V_Y^k(\mathcal{D}_X)\}_{k\in\mathbb{Z}}$ (*along* Y) is a filtration on $\mathcal{D}_X|_Y$ defined by

$$V_Y^k(\mathcal{D}_X) := \bigcap_{j\in\mathbb{Z}} \{P \in \mathcal{D}_X|_Y;\ P\,\mathcal{I}_Y^j \subset \mathcal{I}_Y^{j-k}\}.$$

Let us denote by ϑ the Euler operator. Note that $\vartheta \in V_Y^0(\mathcal{D}_X) \setminus V_Y^{-1}(\mathcal{D}_X)$ and that ϑ can be represented by $\tau\,\partial_\tau$ by admissible coordinates.

3.2 Definition. A coherent $\mathcal{D}_X|_Y$-Module $\mathcal{M}$ is said to be *regular-specializable* (*along* Y) if there exist locally a coherent $\mathcal{O}_X$-sub-Module $\mathcal{M}_0$ of $\mathcal{M}$ and a non-zero polynomial $b(\alpha) \in \mathbb{C}[\alpha]$ such that the following conditions are satisfied:

(1) $\mathcal{M}_0$ generates $\mathcal{M}$ over $\mathcal{D}_X$; that is, $\mathcal{M} = \mathcal{D}_X\,\mathcal{M}_0$;

(2) $b(\vartheta)\,\mathcal{M}_0 \subset (\mathcal{D}_X^{(m)} \cap V_Y^{-1}(\mathcal{D}_X))\,\mathcal{M}_0$, where m is the degree of $b(\alpha)$.

In what follows, we shall omit the phrase "along Y" since Y is fixed.

3.3 Remark. (1) Let $\mathcal{M}$ be a coherent $\mathcal{D}_X|_Y$-Module for which Y is non-characteristic. Then, it is easy to see that $\mathcal{M}$ is regular-specializable.

(2) Kashiwara-Kawai [K-K] proved that every regular-holonomic $\mathcal{D}_X|_Y$-Module is regular-specializable.

3.4 Proposition. *If $\mathcal{M}$ is a regular-specializable $\mathcal{D}_X|_Y$-Module, $\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_Y(\mathcal{O}_X))$ and $\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_Y(\mathcal{O}_X))$ are objects of $\mathbf{D}^b_{\mathbb{C}^\times}(T_Y^*X)$ and $\mathbf{D}^b_{\mathbb{C}^\times}(T_Y X)$ respectively.*

Let $\iota\colon Y \to X$ be the natural inclusion. Then the *induced system,* or the *inverse image* in the sense of $\mathcal{D}$-Modules is defined by $\boldsymbol{D}\iota^*\,\mathcal{M} := \mathcal{O}_Y \overset{\boldsymbol{L}}{\underset{\iota^{-1}\mathcal{O}_X}{\otimes}} \iota^{-1}\,\mathcal{M}$.

For any regular-specializable $\mathcal{D}_X$-Module $\mathcal{M}$, the *nearby-cycle* $\Psi_Y(\mathcal{M})$ of $\mathcal{M}$ and the *vanishing-cycle* $\Phi_Y(\mathcal{M})$ of $\mathcal{M}$ in the theory of $\mathcal{D}$-Modules can be defined. For the definitions of $\Psi_Y(\mathcal{M})$ and $\Phi_Y(\mathcal{M})$, we refer to Laurent [L], Mebkhout [Me]. We shall recall the following two results:

3.5 Proposition (Laurent [L], Mebkhout [Me]). *Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X|_Y$-Module. Then, $\Psi_Y(\mathcal{M})$, $\Phi_Y(\mathcal{M})$ and each cohomology of $\boldsymbol{D}\iota^*\mathcal{M}$ are coherent $\mathcal{D}_Y$-Modules. Moreover, there exists the following distinguished triangle:*

$$\Phi_Y(\mathcal{M}) \xrightarrow{\mathrm{Var}} \Psi_Y(\mathcal{M}) \to \boldsymbol{D}\iota^*\mathcal{M} \xrightarrow{+1} .$$

Here, $\mathrm{Var} := \varphi(\vartheta)\tau$ *with* $\varphi(\zeta) := (e^{2\pi\sqrt{-1}\zeta} - 1)/\zeta$.

3.6 Theorem (Laurent [L]). *Let $\mathcal{C}^{\mathbb{R}}_{Y|X}$ be the sheaf of real holomorphic microfunctions on $T^*_Y X$ as usual. Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X|_Y$-Module. Then, there exists the following isomorphism of distinguished triangles:*

$$\begin{array}{ccccccc}
\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y & \longrightarrow & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \sigma^{-1}\nu_Y(\mathcal{O}_X)) & \longrightarrow & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, {}^t\sigma^{-1}\mathcal{C}^{\mathbb{R}}_{Y|X}) & \xrightarrow{+1} & \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\
\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}\iota^*\mathcal{M}, \mathcal{O}_Y) & \longrightarrow & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{O}_Y) & \longrightarrow & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\Phi_Y(\mathcal{M}), \mathcal{O}_Y) & \xrightarrow{+1} & .
\end{array}$$

3.7 Remark. (1) The isomorphism (the Cauchy-Kovalevskaja type theorem)

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}\iota^*\mathcal{M}, \mathcal{O}_Y) \simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y$$

holds for Fuchsian systems in the sense of Laurent-Monteiro Fernandes [L-MF 1].

(2) Recently Mandai [Man] extended the definition of boundary values to a general Fuchsian differential equation in the complex domain.

4 Boundary Values for Regular-Specializable System

We denote by $\mathcal{O}_X$, $\mathcal{B}_M$ and $\mathcal{C}_M$ the sheaf of *holomorphic functions* on X, of *hyperfunctions* on M and of *microfunctions* on $T^*_M X$ respectively.

4.1 Definition (Oaku [Oa 2], Oaku-Yamazaki [O-Y]). We set:

$$\mathcal{C}_{N|M} := s_{L\pi}^{-1}\,\mu_{\widetilde{M}_N}(\boldsymbol{R}j_{L*}\,\widetilde{p}_L^{-1}\,i_L^!\,\mathcal{O}_X) \otimes or_{M/X}[n+1] .$$

We can regard $\mathcal{C}_{N|M}$ as a microlocalization of $\nu_N(\mathcal{B}_M)$:

4.2 Proposition. (1) *$\mathcal{C}_{N|M}$ is concentrated in degree zero; that is, $\mathcal{C}_{N|M}$ is regarded as a sheaf on $T^*_{T_N M} T_Y L$. Further $\mathcal{C}_{N|M}|_{T_N M} = \nu_N(\mathcal{B}_M)$ holds.*

(2) *There exists the following exact sequence on $T_N M$:*

$$0 \to \nu_Y(\mathcal{B}\mathcal{O}_L)|_{T_N M} \to \nu_N(\mathcal{B}_M) \to \dot{\pi}_{N|M*}\,\mathcal{C}_{N|M} \to 0.$$

Here $\mathcal{B}\mathcal{O}_L := \mathcal{H}^1_L(\mathcal{O}_X) \otimes or_{L/X}$ is the sheaf of hyperfunctions with holomorphic parameters on L. Note that $\nu_Y(\mathcal{B}\mathcal{O}_L)$ is concentrated in degree zero.

4.3 Definition. Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X|_Y$-Module. By Proposition 3.4, $\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ satisfies the assumption of Theorem 2.1. Thus, by Definition 2.2 and Theorem 3.6, we define:

$$\beta\colon f_\pi^{-1}\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}) \to f_\pi^{-1}\tau_{Y\pi}^{-1}\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{C}_N).$$

4.4 Theorem. (1) *The morphism β gives a monomorphism*

$$\beta^0\colon f_\pi^{-1}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}) \rightarrowtail f_\pi^{-1}\tau_{Y\pi}^{-1}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{C}_N).$$

(2) *The restriction of β^0 to the zero-section T_NM^+ coincides with the boundary value morphism in the sense of Monteiro Fernandes* [MF 1].

4.5 Definition. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X|_Y$-Module. Then we say $\mathcal{M}$ is *near-hyperbolic* at $x_0 \in N$ (in dt-codirection) if $\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ is near-hyperbolic in the sense of Definition 2.3. Here, we remark that $\mathrm{SS}(\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) = \mathrm{char}(\mathcal{M})$.

The following theorem is a direct consequence of Theorem 2.4:

4.6 Theorem. *Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X|_Y$-Module. Assume that $\mathcal{M}$ is near-hyperbolic at $x_0 \in N$. Then, for any $p^* \in T^*_{T_NM^+}T_YL^+$*

$$\beta\colon \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M})_{p^*} \to \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{C}_N)_{\tau_{Y\pi}(p^*)}$$

is an isomorphism.

4.7 Remark. Let $\mathcal{C}^F_{N|M}$ be the sheaf of F-mild microfunctions on $T^*_{T_NM}T_YL$, and set $\widetilde{\mathcal{C}}^A_{N|M} := \mathcal{H}^n(\mu_N(\mathcal{O}_X|_Y)) \otimes or_{N/Y}$ (see Oaku [Oa 1], [Oa 2], and Oaku-Yamazaki [O-Y]). Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X|_Y$-Module. Set $\mathcal{M}_Y := \mathcal{H}^0(\boldsymbol{D}\iota^*\mathcal{M}) = \mathcal{O}_Y \underset{\iota^{-1}\mathcal{O}_X}{\otimes} \iota^{-1}\mathcal{M}$. By the argument in Oaku-Yamazaki [O-Y] we have the following commutative diagram:

$$\begin{array}{ccccc}
f_\pi^{-1}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}^F_{N|M}) & \rightarrowtail & f_\pi^{-1}\tau_{Y\pi}^{-1}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \widetilde{\mathcal{C}}^A_{N|M}) & \xrightarrow{\sim} & f_\pi^{-1}\tau_{Y\pi}^{-1}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N) \\
\downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow \\
f_\pi^{-1}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}) & \rightarrowtail & f_\pi^{-1}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \widetilde{\mathcal{C}}_{N|M}) & \xrightarrow{\sim} & f_\pi^{-1}\tau_{Y\pi}^{-1}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{C}_N),
\end{array}$$

that is, the boundary value morphism

$$\gamma^F\colon f_\pi^{-1}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}^F_{N|M}) \rightarrowtail f_\pi^{-1}\tau_{Y\pi}^{-1}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N)$$

and β^0 are compatible. In particular, if Y is non-characteristic for $\mathcal{M}$, then it is known that $\Psi_Y(\mathcal{M}) \xrightarrow{\sim} \boldsymbol{D}\iota^*\mathcal{M} \simeq \mathcal{M}_Y$ and by Oaku [Oa 2] (cf. Oaku-Yamazaki [O-Y]) we have:

$$\widetilde{\gamma}_{N|M}\colon \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \widetilde{\mathcal{C}}_{N|M}) \xrightarrow{\sim} \tau_{Y\pi}^{-1}\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N).$$

In this case we see that β^0 is equivalent to the non-characteristic boundary value morphism (see Kataoka [Kat] and Oaku [Oa 2]). In particular, the restriction of β^0 to the zero-section $T_N M^+$ is equivalent to Komatsu-Kawai [Ko-K] and Schapira [Sc 1]. Further, if Y is non-characteristic for $\mathcal{M}$ and $\pm dt \in T_N^* M$ is hyperbolic for $\mathcal{M}$, then the nearly-hyperbolic condition is satisfied and β is an isomorphism.

4.8 Example. Assume that $X = \mathbb{C}^{n+1}$ and so on by an admissible coordinate system.

(1) Let $b(\alpha)$ be a non-zero polynomial with degree m, and $Q \in \mathcal{D}_X^{(m)} \cap V_Y^{-1}(\mathcal{D}_X)$. Set $\mathcal{M} := \mathcal{D}_X / \mathcal{D}_X\, (b(\vartheta) + Q)$. Then $\mathcal{M}$ is regular-specializable. Assume that $b(\alpha) = \prod_{j=1}^{\mu} (\alpha - \alpha_j)^{\nu_j}$ ($\alpha_i - \alpha_j \notin \mathbb{Z}$ for $1 \leqslant i \neq j \leqslant \mu$, note that $\sum_{j=1}^{\mu} \nu_j = m$). Then a direct calculation shows that $\Psi_Y(\mathcal{M}) \simeq \mathcal{D}_Y^{\oplus m}$, and β^0 is equivalent to γ in Oaku [Oa 2]: Let $p^* = (x_0, t_0; \sqrt{-1}\langle \xi_0, dx\rangle)$ be a point of $T^*_{T_N M^+} T_Y L^+$, and $f(x,t)$ a germ of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M})$ at p^*. Then, we can see that $f(x,t)$ has a defining function

$$F(z,\tau) = \sum_{j=1}^{\mu} \sum_{k=1}^{\nu_j} F_{jk}(z,\tau)\, \tau^{\alpha_j} (\log \tau)^{k-1}.$$

Here each $F_{jk}(z,\tau)$ is holomorphic on a neighborhood of $\{(z,0) \in X;\ |x_0 - z| < \varepsilon,\ \operatorname{Im} z \in \Gamma\}$ with a positive constant ε and an open convex cone Γ such that $\xi_0 \in \operatorname{Int}(\Gamma^\circ)$ (the interior of the dual cone Γ° of Γ). Then, $\beta^0(f)$ is equivalent to $\{\mathrm{sp}_N(F_{jk}(x + \sqrt{-1}\,\Gamma\, 0, 0));\ 1 \leqslant k \leqslant \nu_j,\ 1 \leqslant j \leqslant \mu\}$. Moreover, if the principal symbol of $b(\vartheta) + Q$ is written as $\tau^m P(z,\tau; z^*, \tau^*)$ for a hyperbolic polynomial P at dt-codirection, the nearly-hyperbolic condition is satisfied. Note that this operator is a special case of Fuchsian hyperbolic operators due to Tahara [T].

(2) Take an operator $A(z;\partial_z) \in \mathcal{D}_Y^{(1)}$ at the origin and set $A^0 := \mathrm{id}$ and $A^{(j)} := \dfrac{1}{j!} A \circ A^{(j-1)} \in \mathcal{D}_Y^{(j)}$ for $j \geqslant 1$. Let $p^* = (0,1; \sqrt{-1}\langle \xi, dx\rangle)$ be a point of $T^*_{T_N M^+} T_Y L^+$ and set $p_0 := (0; \sqrt{-1}\langle \xi, dx\rangle) \in T_N^* Y$. Set $P := (\vartheta - \alpha_1)(\vartheta - \alpha_2) - \tau A(z;\partial_z)\vartheta \in \mathcal{D}_X|_Y$, where $(\alpha_1, \alpha_2) \in \mathbb{C}^{\oplus 2}$. Consider $\mathcal{M} := \mathcal{D}_X / \mathcal{D}_X P = \mathcal{D}_X u$, where $u := 1 \bmod P$. Let $f(x,t)$ be a germ of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M})$ at p^*. Then:

(i) If $(\alpha_1, \alpha_2) = (-1, 0)$, then

$$\Phi_Y(\mathcal{M}) = \frac{V_Y^0(\mathcal{D}_X)u + V_Y^1(\mathcal{D}_X)(\vartheta+1)u}{V_Y^{-1}(\mathcal{D}_X)u + V_Y^0(\mathcal{D}_X)(\vartheta+1)u} = \mathcal{D}_Y[u] + \mathcal{D}_Y[\partial_\tau(\vartheta+1)u] \simeq \mathcal{D}_Y^{\oplus 2},$$

$$\Psi_Y(\mathcal{M}) = \frac{V_Y^{-1}(\mathcal{D}_X)u + V_Y^0(\mathcal{D}_X)(\vartheta+1)u}{V_Y^{-2}(\mathcal{D}_X)u + V_Y^{-1}(\mathcal{D}_X)(\vartheta+1)u} = \mathcal{D}_Y[\tau u] + \mathcal{D}_Y[(\vartheta+1)u] \simeq \mathcal{D}_Y^{\oplus 2},$$

and Var: $([u], [\partial_\tau(\vartheta - 1)u]) \mapsto ([\tau u], 0)$. Hence $\mathcal{M}_Y \simeq \mathcal{D}_Y[(\vartheta+1)u] \simeq \mathcal{D}_Y$. In this case $f(x,t)$ has the following defining function:

$$F(z,\tau) = U_0(z) + \frac{U_{-1}(z)}{\tau} - \sum_{j=1}^{\infty} \frac{A^{(j)} U_{-1}(z)}{j-1} \tau^{j-1} - A U_{-1}(z) \log \tau,$$

and $\beta^0(f(x,t))$ is given by $\{\mathrm{sp}_N(U_i)(x)\}_{i=-1,0}$ at p_0. If $f(x,t)$ is F-mild at p_0, then $U_{-1}(z)=0$ and $\gamma^F(f(x,t))=\{f(x,+0)\}=\{\mathrm{sp}_N(U_0)(x)\}$.

(ii) If $(\alpha_1,\alpha_2)=(0,1)$, then:

$$\begin{aligned}\varPhi_Y(\mathcal{M}) &= \frac{V_Y^1(\mathcal{D}_X)u+V_Y^2(\mathcal{D}_X)\vartheta u}{V_Y^0(\mathcal{D}_X)u+V_Y^1(\mathcal{D}_X)\vartheta u} = \mathcal{D}_Y\,[\partial_\tau u]+\mathcal{D}_Y\,[\partial_\tau{}^2\vartheta u]\simeq \mathcal{D}_Y^{\oplus 2},\\ \varPsi_Y(\mathcal{M}) &= \frac{V_Y^0(\mathcal{D}_X)u+V_Y^1(\mathcal{D}_X)\vartheta u}{V_Y^{-1}(\mathcal{D}_X)u+V_Y^0(\mathcal{D}_X)\vartheta u} = \mathcal{D}_Y\,[u]+\mathcal{D}_Y\,[\partial_\tau\vartheta u]\simeq \mathcal{D}_Y^{\oplus 2},\end{aligned}$$

and $\mathrm{Var}\,[\partial_\tau u]=\mathrm{Var}\,[\partial_\tau{}^2\vartheta u]=0$. Hence $\mathcal{M}_Y\simeq\mathcal{D}_Y\,[u]+\mathcal{D}_Y\,[\partial_\tau\vartheta u]\simeq\mathcal{D}_Y^{\oplus 2}$. In this case $f(x,t)$ has the following defining function:

$$F(z,\tau)=U_0(z)+\sum_{j=0}^{\infty}\frac{A^{(j)}U_1(z)}{j+1}\,\tau^{j+1},$$

and $f(x,t)$ is always F-mild. Hence $\beta^0(f(x,t))$ at p_0 coincides with $\gamma^F(f(x,t))=\{\partial_t{}^i f(x,+0)\}_{i=0,1}=\{\mathrm{sp}_N(U_i)(x)\}_{i=0,1}$ (if $\tau\neq 0$, $\mathcal{M}$ is isomorphic to $\mathcal{D}_X/\,\mathcal{D}_X(\partial_\tau^2-A(z;\partial_z)\,\partial_\tau)$ for which Y is non-characteristic).

(iii) If $(\alpha_1,\alpha_2)=(1,1)$, then

$$\begin{aligned}\varPhi_Y(\mathcal{M}) &= \frac{V_Y^2(\mathcal{D}_X)u}{V_Y^1(\mathcal{D}_X)u} = \mathcal{D}_Y\,[\partial_\tau{}^2\,u]+\mathcal{D}_Y\,[\partial_\tau{}^2(\vartheta-1)u]\simeq \mathcal{D}_Y^{\oplus 2},\\ \varPsi_Y(\mathcal{M}) &= \frac{V_Y^1(\mathcal{D}_X)u}{V_Y^0(\mathcal{D}_X)u} = \mathcal{D}_Y\,[\partial_\tau u]+\mathcal{D}_Y\,[\partial_\tau(\vartheta-1)u]\simeq \mathcal{D}_Y^{\oplus 2}.\end{aligned}$$

and Var: $([\partial_\tau{}^2\,u],[\partial_\tau{}^2(\vartheta-1)u])\mapsto(2\pi\sqrt{-1}\,[\partial_\tau(\vartheta-1)u],0)$. Hence $\mathcal{M}_Y\simeq\mathcal{D}_Y\,[\partial_\tau u]\simeq\mathcal{D}_Y$. In this case $f(x,t)$ has the following defining function:

$$F(z,\tau)=\sum_{j=0}^{\infty}A^{(j)}U_0(z)\,\tau^{j+1}-\sum_{j=1}^{\infty}\sum_{k=1}^{j}\frac{A^{(j)}U_1(z)}{k}\,\tau^{j+1}+\sum_{j=0}^{\infty}A^{(j)}U_1(z)\,\tau^{j+1}\log\tau\,,$$

and $\beta^0(f(x,t))$ is given by $\{\mathrm{sp}_N(U_i)(x)\}_{i=0,1}$ at p_0. If $f(x,t)$ is F-mild at p_0, then $U_0(z)=0$ and $\gamma^F(f(x,t))=\{\partial_t\,f(x,+0)\}=\{\mathrm{sp}_N(U_1)(x)\}$.

(iv) If $(\alpha_1,\alpha_2)=(1,2)$, then:

$$\begin{aligned}\varPhi_Y(\mathcal{M}) &= \frac{V_Y^2(\mathcal{D}_X)u+V_Y^3(\mathcal{D}_X)(\vartheta-1)u}{V_Y^1(\mathcal{D}_X)u+V_Y^2(\mathcal{D}_X)(\vartheta-1)u} = \mathcal{D}_Y\,[\partial_\tau{}^2\,u]+\mathcal{D}_Y\,[\partial_\tau{}^3(\vartheta-1)u]\simeq \mathcal{D}_Y^{\oplus 2},\\ \varPsi_Y(\mathcal{M}) &= \frac{V_Y^1(\mathcal{D}_X)u+V_Y^2(\mathcal{D}_X)(\vartheta-1)u}{V_Y^0(\mathcal{D}_X)u+V_Y^1(\mathcal{D}_X)(\vartheta-1)u} = \mathcal{D}_Y\,[\partial_\tau u]+\mathcal{D}_Y\,[\partial_\tau{}^2(\vartheta-1)u]\simeq \mathcal{D}_Y^{\oplus 2},\end{aligned}$$

and Var: $([\partial_\tau{}^2\,u],[\partial_\tau{}^3(\vartheta-1)u])\mapsto(0,2A\,[\partial_\tau u])$. Hence

$$\mathcal{M}_Y\simeq\frac{\mathcal{D}_Y\,[\partial_\tau u]+\mathcal{D}_Y\,[\partial_\tau{}^2(\vartheta-1)u]}{\mathcal{D}_Y\,A\,[\partial_\tau u]}\,.$$

In this case $f(x,t)$ has the following defining function:

$$F(z,\tau)=\sum_{j=0}^{\infty}A^{(j)}U_2(z)\,\tau^{j+2}+U_1(z)\,\tau-\sum_{j=2}^{\infty}\sum_{k=1}^{j-1}\frac{jA^{(j)}U_1(z)}{k}\,\tau^{j+1}$$
$$+\Big(\sum_{j=0}^{\infty}(j+1)A^{(j+1)}U_1(z)\,\tau^{j}\Big)\,\tau^2\log\tau,$$

and $\beta^0\big(f(x,t)\big)$ is given by $\{\mathrm{sp}_N(U_i)(x)\}_{i=1,2}$ at p_0. $f(x,t)$ is F-mild under the condition that $AU_1(z)=0$, and in this case $\gamma^F\big(f(x,t)\big)$ at p_0 is given by $\gamma^F\big(f_3(x,t)\big)=\{\partial_t{}^i f(x,+0)\}_{i=1,2}=\{\mathrm{sp}_N(U_1)(x),2\,\mathrm{sp}_N(U_2)(x)\}$ with $A\partial_t f(x,+0)=A\,\mathrm{sp}_N(U_1)(x)=0$.

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